Vector analysis 3

Frames 1 to 40

Learning outcomes

When you have completed this Programme you will be able to:

- Derive the family of curves of constant coordinates for curvilinear coordinates
- Derive unit base vectors and scale factors in orthogonal curvilinear coordinates
- Obtain the element of arc ds and the element of volume dV in orthogonal curvilinear coordinates
- Obtain expressions for the operators grad, div and curl in orthogonal curvilinear coordinates

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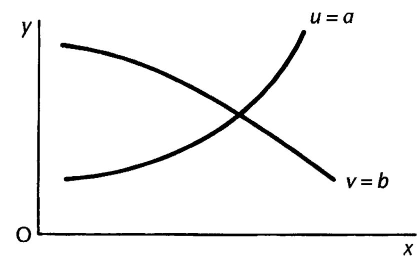
This short Programme is an extension of the two previous ones and may not be required for all courses. It can well be bypassed without adversely affecting the rest of the work.

Curvilinear coordinates

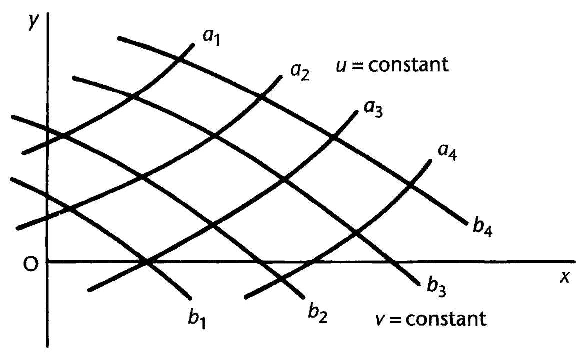
Let us consider two variables u and v, each of which is a function of x and y

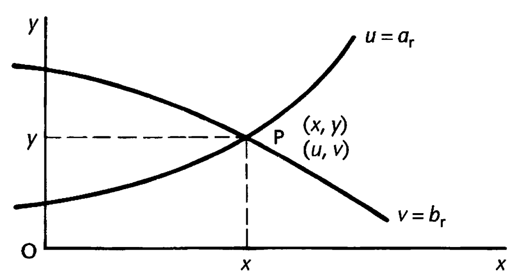
i.e. $u = f(x, y)$
$v = g(x, y)$



If u and v are each assigned a constant value a and b, the equations will, in general, define two intersecting curves.

If u and v are each given several such values, the equations define a network of curves covering the x–y plane.





A pair of curves $u = a_r$ and $v = b_r$ pass through each point in the plane. Hence, any point in the plane can be expressed in *rectangular coordinates* (x, y) or in *curvilinear coordinates* (u, v).

Let us see how this works out in an example, so move on

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Example 1

Let us consider the case where $u = xy$ and $v = x^2 - y$.

(a) With $u = xy$, if we put $u = 4$, then $y = \dfrac{4}{x}$ and we can plot y against x to obtain the relevant curve.

Similarly, putting $u = 8, 16, 32, \ldots$ we can build up a family of curves, all of the pattern $u = xy$.

x		0·5	1·0	2·0	3·0	4·0
y	$u = 4$	8	4	2	1·33	1·0
	$u = 8$	16	8	4	2·67	2
	$u = 16$	32	16	8	5·33	4
	$u = 32$	64	32	16	10·67	8

If we plot these on graph paper between $x = 0$ and $x = 4$ with a range of y from $y = 0$ to $y = 20$, we obtain

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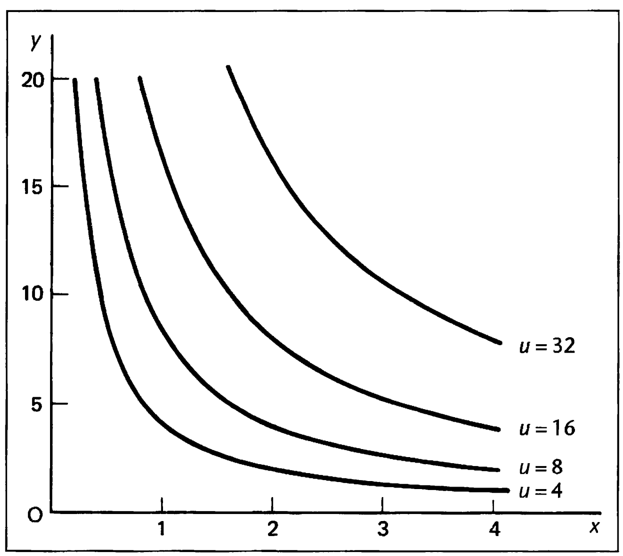

Note that each graph is labelled with its individual u-value.

(b) With $v = x^2 - y$, we proceed in just the same way. We rewrite the equation as $y = x^2 - v$; assign values such as 8, 4, 0, −4, −8, −12, −16, ... to v; and draw the relevant curve in each case. If we do that for $x = 0$ to $x = 4$ and limit the y-values to the range $y = 0$ to $y = 20$, we obtain the family of curves

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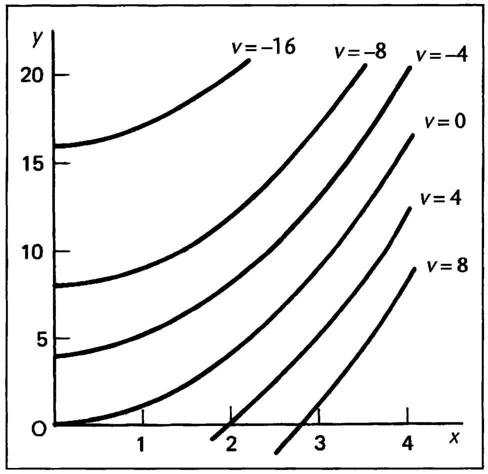

The table of function values is as follows.

x		0	1	2	3	4
y	$v = 8$	-8	-7	-4	1	8
	$v = 4$	-4	-3	0	5	12
	$v = 0$	0	1	4	9	16
	$v = -4$	4	5	8	13	20
	$v = -8$	8	9	12	17	24
	$v = -12$	12	13	16	21	28
	$v = -16$	16	17	20	25	32

Note again that we label each graph with its own v-value.

This again is a family of curves with the common pattern $v = x^2 - y$, the members being distinguished from each other by the value assigned to v in each case.

Now we draw both sets of curves on a common set of x–y axes, taking

the range of x from $x = 0$ to $x = 4$

and the range of y from $y = 0$ to $y = 20$.

It is worthwhile taking a little time over it – and good practice!

When you have the complete picture, move on to the next frame

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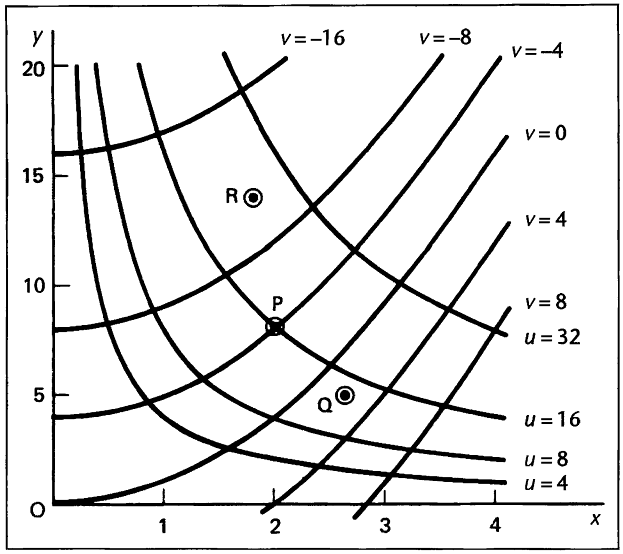

The position of any point in the plane can now be stated in two ways. For example, the point P has Cartesian rectangular coordinates $x = 2$, $y = 8$. It can also be stated in curvilinear coordinates $u = 16$, $v = -4$, for it is at the point of intersection of the two curves corresponding to $u = 16$ and $v = -4$.

Likewise, for the point Q, the position in rectangular coordinates is $x = 2{\cdot}65$, $y = 5{\cdot}0$ and for its position in curvilinear coordinates we must estimate it within the network. Approximate values are $u = 13$, $v = 2$.

Similarly, the curvilinear coordinates of R ($x = 1{\cdot}8$, $y = 14$) are approximately

$u = \ldots\ldots\ldots\ldots; \quad v = \ldots\ldots\ldots\ldots$

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$$\boxed{u = 26; \quad v = -11}$$

Their actual values are in fact $u = 25{\cdot}2$ and $v = -10{\cdot}76$.

Now let us deal with another example.

7

Example 2

If $u = x^2 + 2y$ and $v = y - (x+1)^2$, these can be rewritten as $y = \frac{1}{2}(u - x^2)$ and $y = v + (x+1)^2$. We can now plot the family of curves, say between $x = 0$ and $x = 4$, with $u = 5(5)30$ and $v = -20(5)5$, i.e. values of u from 5 to 30 at intervals of 5 units and values of v from -20 to 5 at intervals of 5 units.

The resulting network is easily obtained and appears as

. . . .

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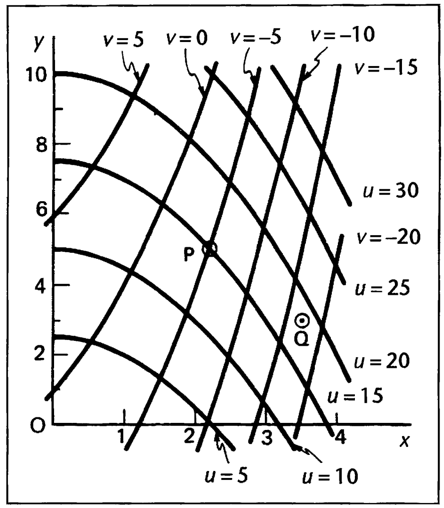

For P, the rectangular coordinates are ($x = 2{\cdot}18$, $y = 5{\cdot}1$) and the curvilinear coordinates are ($u = 15$, $v = -5$).
For Q, the rectangular coordinates are ....
and the curvilinear coordinates are ....

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Q: ($x = 3{\cdot}5$, $y = 3{\cdot}0$); ($u = 18{\cdot}5$, $v = -17$)

Orthogonal curvilinear coordinates

If the coordinate curves for u and v forming the network cross at right angles, the system of coordinates is said to be *orthogonal*. The test for orthogonality is given by the dot product of the vectors formed from the partial derivatives. This is, if

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 \text{ then } u \text{ and } v \text{ are orthogonal.}$$

Example 3

Given the curvilinear coordinates u and v where $u = xy$ and $v = x^2 - y^2$ then

u and v form a coordinate system that is ....

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$\boxed{\text{orthogonal}}$

Because

$$u = xy \text{ so } \frac{\partial u}{\partial x} = y \text{ and } \frac{\partial u}{\partial y} = x,\ v = x^2 - y^2 \text{ so } \frac{\partial v}{\partial x} = 2x \text{ and } \frac{\partial v}{\partial y} = -2y.$$

Then $\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y} = 2xy - 2xy = 0$ and so u and v form a coordinate system that is orthogonal.

Example 4

Given the curvilinear coordinates u and v where $u = x^2 + 2y$ and $v = y - (x+1)^2$ then

u and v form a coordinate system that is ....

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$\boxed{\text{not orthogonal}}$

Because

$$u = x^2 + 2y \text{ so } \frac{\partial u}{\partial x} = 2x \text{ and } \frac{\partial u}{\partial y} = 2,\ v = y - (x+1)^2 \text{ so } \frac{\partial v}{\partial x} = -2(x+1)$$

and $\dfrac{\partial v}{\partial y} = 1$.

Then

$\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial y} = -4x(x+1) + 2 \neq 0$ and so u and v form a coordinate system that is not orthogonal.

Let us extend these ideas to three dimensions. Move on

Orthogonal coordinate systems in space

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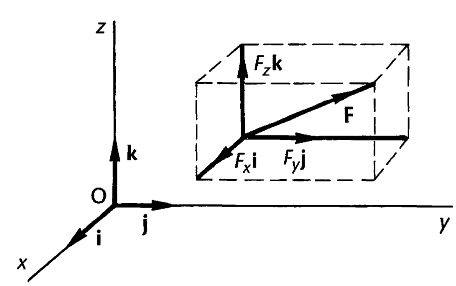
Any vector $\mathbf{F}$ can be expressed in terms of its components in three mutually perpendicular directions, which have normally been the directions of the coordinate axes, i.e.

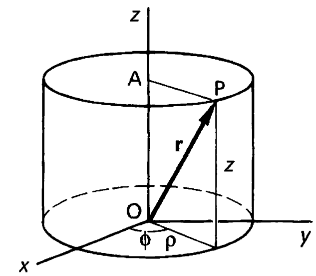
$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$



where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors parallel to the x, y, z axes respectively. ▶

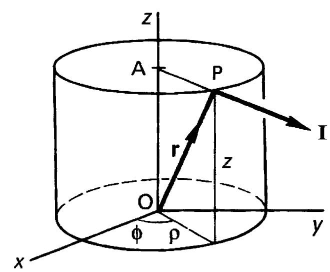
Situations can arise, however, where the directions of the unit vectors do not remain fixed, but vary from point to point in space according to prescribed conditions. Examples of this occur in cylindrical and spherical polar coordinates, with which we are already familiar.

1 *Cylindrical polar coordinates* (ρ, ϕ, z)
Let P be a point with cylindrical coordinates (ρ, ϕ, z) as shown. The position of P is a function of the three variables ρ, ϕ, z



(a) If ϕ and z remain constant and ρ varies, then P will move out along AP by an amount $\dfrac{\partial \mathbf{r}}{\partial \rho}$ and the unit vector $\mathbf{I}$ in this direction will be given by

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial \rho} \Bigg/ \left|\frac{\partial \mathbf{r}}{\partial \rho}\right|$$



(b) If, instead, ρ and z remain constant and ϕ varies, P will move

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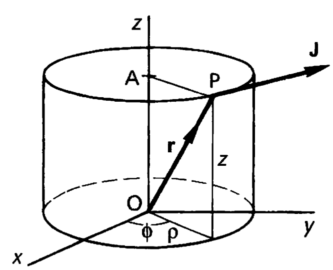
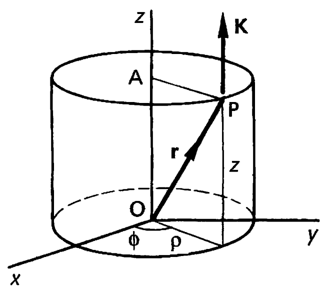
round the circle with AP as radius



$\dfrac{\partial \mathbf{r}}{\partial \phi}$ is therefore a vector along the tangent to the circle at P and the unit vector $\mathbf{J}$ at P will be given by

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \phi} \Bigg/ \left|\frac{\partial \mathbf{r}}{\partial \phi}\right|$$

▶



(c) Finally, if ρ and ϕ remain constant and z increases, the vector $\dfrac{\partial \mathbf{r}}{\partial z}$ will be parallel to the z-axis and the unit vector $\mathbf{K}$ in this direction will be given by

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial z} \Big/ \left|\frac{\partial \mathbf{r}}{\partial z}\right|$$

Putting our three unit vectors on to one diagram, we have

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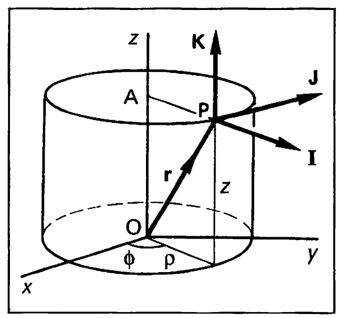

Note that $\mathbf{I}, \mathbf{J}, \mathbf{K}$ are mutually perpendicular and form a right-handed set. But note also that, unlike the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the Cartesian system, the unit vectors $\mathbf{I}, \mathbf{J}, \mathbf{K}$, or *base vectors* as they are called, are not fixed in directions, but change as the position of P changes.

So we have, for cylindrical polar coordinates

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial \rho} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \rho}\right|$$

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \phi} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \phi}\right|$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial z} \Big/ \left|\frac{\partial \mathbf{r}}{\partial z}\right|$$

If $\mathbf{F}$ is a vector associated with P, then $\mathbf{F(r)} = F_\rho \mathbf{I} + F_\phi \mathbf{J} + F_z \mathbf{K}$ where F_ρ, F_ϕ, F_z are the components of $\mathbf{F}$ in the directions of the unit base vectors $\mathbf{I}, \mathbf{J}, \mathbf{K}$.

Now let us attend to spherical coordinates in the same way.

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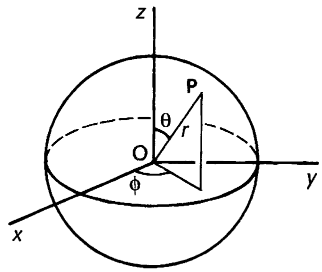
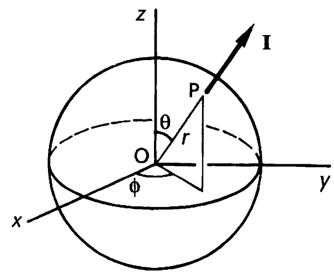
2 *Spherical polar coordinates* (r, θ, ϕ)

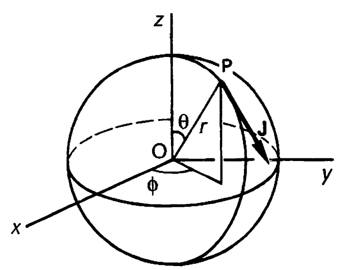


P is a function of the three variables r, θ, ϕ.



(a) If θ and ϕ remain constant and r increases, P moves outwards in the direction OP. $\dfrac{\partial \mathbf{r}}{\partial r}$ is thus a vector normal to the surface of the sphere at P and the unit vector $\mathbf{I}$ in that direction is therefore

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial r} \Big/ \left|\frac{\partial \mathbf{r}}{\partial r}\right|$$



(b) If r and ϕ remain constant and θ increases, P will move along the 'meridian' through P, i.e. $\dfrac{\partial \mathbf{r}}{\partial \theta}$ is a tangent vector to this circle at P and the unit vector $\mathbf{J}$ is given by

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \theta} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \theta}\right|$$

(c) If r and θ remain constant and ϕ increases, P will move

....

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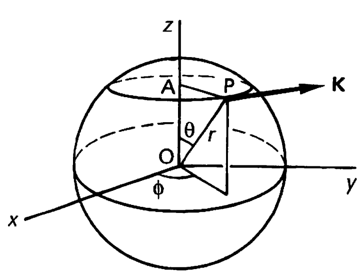
along the circle through P perpendicular to the z-axis



$\dfrac{\partial \mathbf{r}}{\partial \phi}$ is therefore a tangent vector at P and the unit vector $\mathbf{K}$ in this direction is given by

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \phi} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \phi}\right|$$

So, putting the three results on one diagram, we have ....

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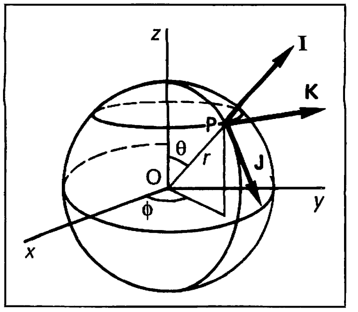

Once again, the three unit vectors at P (base vectors) are mutually perpendicular and form a right-handed set. Their directions in space, however, change as the position of P changes.

A vector **F** associated with P can therefore be expressed as $\mathbf{F} = F_r\mathbf{I} + F_\theta\mathbf{J} + F_\phi\mathbf{K}$ where F_r, F_θ, F_ϕ are the components of **F** in the directions of the base vectors **I**, **J**, **K**.

Both cylindrical and spherical polar coordinate systems are

....

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orthogonal

Scale factors

Collecting the recent results together, we have:

1 For cylindrical polar coordinates, the unit base vectors are

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial \rho} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \rho}\right| = \frac{1}{h_\rho}\frac{\partial \mathbf{r}}{\partial \rho} \qquad \text{where } h_\rho = \left|\frac{\partial \mathbf{r}}{\partial \rho}\right|$$

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \phi} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = \frac{1}{h_\phi}\frac{\partial \mathbf{r}}{\partial \phi} \qquad \text{where } h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right|$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial z} \Big/ \left|\frac{\partial \mathbf{r}}{\partial z}\right| = \frac{1}{h_z}\frac{\partial \mathbf{r}}{\partial z} \qquad \text{where } h_z = \left|\frac{\partial \mathbf{r}}{\partial z}\right|$$

2 For spherical polar coordinates, the unit base vectors are

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial r} \Big/ \left|\frac{\partial \mathbf{r}}{\partial r}\right| = \frac{1}{h_r}\frac{\partial \mathbf{r}}{\partial r} \qquad \text{where } h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right|$$

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \theta} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = \frac{1}{h_\theta}\frac{\partial \mathbf{r}}{\partial \theta} \qquad \text{where } h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right|$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \phi} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = \frac{1}{h_\phi}\frac{\partial \mathbf{r}}{\partial \phi} \qquad \text{where } h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right|$$

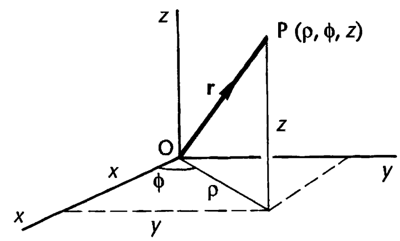
In each case, h is called the *scale factor*.

Move on

19 Scale factors for coordinate systems

1 *Rectangular coordinates* (x, y, z)
With rectangular coordinates, $h_x = h_y = h_z = 1$.

2 *Cylindrical coordinates* (ρ, ϕ, z)



$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

$$\therefore\ \mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j} + z\,\mathbf{k}$$

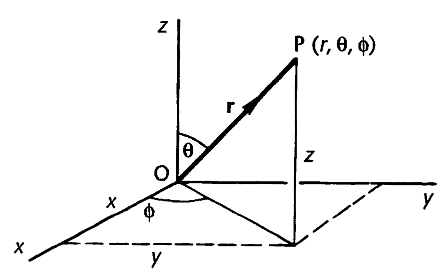
$$\mathbf{I} = \frac{\partial\mathbf{r}}{\partial\rho}\bigg/\left|\frac{\partial\mathbf{r}}{\partial\rho}\right| = \frac{1}{h_\rho}\frac{\partial\mathbf{r}}{\partial\rho} \qquad h_\rho = \left|\frac{\partial\mathbf{r}}{\partial\rho}\right| = |\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}|$$
$$= (\cos^2\phi + \sin^2\phi)^{1/2} = 1$$
$$\therefore\ h_\rho = 1$$

$$\mathbf{J} = \frac{\partial\mathbf{r}}{\partial\phi}\bigg/\left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = \frac{1}{h_\phi}\frac{\partial\mathbf{r}}{\partial\phi} \qquad h_\phi = \left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = |-\rho\sin\phi\,\mathbf{i} + \rho\cos\phi\,\mathbf{j}|$$
$$= (\rho^2\sin^2\phi + \rho^2\cos^2\phi)^{1/2} = \rho$$
$$\therefore\ h_\phi = \rho$$

$$\mathbf{K} = \frac{\partial\mathbf{r}}{\partial z}\bigg/\left|\frac{\partial\mathbf{r}}{\partial z}\right| = \frac{1}{h_z}\frac{\partial\mathbf{r}}{\partial z} \qquad h_z = \left|\frac{\partial\mathbf{r}}{\partial z}\right| = |\mathbf{k}| = 1$$
$$\therefore\ h_z = 1$$

$$\therefore\ h_\rho = 1;\ h_\phi = \rho;\ h_z = 1$$

3 *Spherical coordinates* (r, θ, ϕ)



$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\therefore\ \mathbf{r} = r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$$

Then working as before

$h_r = \ldots\ldots\ldots\ldots$; $h_\theta = \ldots\ldots\ldots\ldots$; $h_\phi = \ldots\ldots\ldots\ldots$

$$\boxed{h_r = 1; \quad h_\theta = r; \quad h_\phi = r\sin\theta}$$

Because

$$\mathbf{r} = r\sin\theta\cos\phi\,\mathbf{i} + r\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$$

$$\mathbf{I} = \frac{\partial\mathbf{r}}{\partial r}\bigg/\left|\frac{\partial\mathbf{r}}{\partial r}\right| = \frac{1}{h_r}\frac{\partial\mathbf{r}}{\partial r}$$

$$\begin{aligned} h_r = \left|\frac{\partial\mathbf{r}}{\partial r}\right| &= |\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}| \\ &= (\sin^2\theta\cos^2\phi + \sin^2\theta\ \sin^2\phi + \cos^2\theta)^{1/2} \\ &= (\sin^2\theta + \cos^2\theta)^{1/2} = 1 \end{aligned}$$

$$\therefore\ h_r = 1$$

$$\mathbf{J} = \frac{\partial\mathbf{r}}{\partial\theta}\bigg/\left|\frac{\partial\mathbf{r}}{\partial\theta}\right| = \frac{1}{h_\theta}\frac{\partial\mathbf{r}}{\partial\theta}$$

$$\begin{aligned} h_\theta = \left|\frac{\partial\mathbf{r}}{\partial\theta}\right| &= |\,r\ \cos\theta\ \cos\phi\,\mathbf{i} + r\ \cos\theta\ \sin\phi\,\mathbf{j} - r\ \sin\theta\,\mathbf{k}\,| \\ &= (r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta)^{1/2} \\ &= (r^2\cos^2\theta + r^2\sin^2\theta)^{1/2} = r \end{aligned}$$

$$\therefore\ h_\theta = r$$

$$\mathbf{K} = \frac{\partial\mathbf{r}}{\partial\phi}\bigg/\left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = \frac{1}{h_\phi}\frac{\partial\mathbf{r}}{\partial\phi}$$

$$\begin{aligned} h_\phi = \left|\frac{\partial\mathbf{r}}{\partial\phi}\right| &= |-r\sin\theta\sin\phi\,\mathbf{i} + r\sin\theta\cos\phi\,\mathbf{j}\,| \\ &= (r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi)^{1/2} \\ &= (r^2\sin^2\theta)^{1/2} = r\sin\theta \end{aligned}$$

$$\therefore\ h_\phi = r\sin\theta$$

$$\therefore\ h_r = 1; \quad h_\theta = r; \quad h_\phi = r\sin\theta$$

So: (a) for cylindrical coordinates

$$\mathbf{I} = \frac{\partial\mathbf{r}}{\partial\rho}; \quad \mathbf{J} = \frac{1}{\rho}\frac{\partial\mathbf{r}}{\partial\phi}; \quad \mathbf{K} = \frac{\partial\mathbf{r}}{\partial z}$$

(b) for spherical coordinates

$$\mathbf{I} = \frac{\partial\mathbf{r}}{\partial r}; \quad \mathbf{J} = \frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}; \quad \mathbf{K} = \frac{1}{r\sin\theta}\frac{\partial\mathbf{r}}{\partial\phi}$$

General curvilinear coordinate system (u, v, w)

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Any system of coordinates can be treated in like manner to obtain expressions for the appropriate unit vectors **I**, **J**, **K**.

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial u} \bigg/ \left|\frac{\partial \mathbf{r}}{\partial u}\right|; \quad \mathbf{J} = \frac{\partial \mathbf{r}}{\partial v} \bigg/ \left|\frac{\partial \mathbf{r}}{\partial v}\right|; \quad \mathbf{K} = \frac{\partial \mathbf{r}}{\partial w} \bigg/ \left|\frac{\partial \mathbf{r}}{\partial w}\right|$$

These unit vectors are not always at right angles to each other.

If they are mutually perpendicular, the coordinate system is

....

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orthogonal

Unit vectors **I**, **J**, **K** are orthogonal if

$$\mathbf{I} \cdot \mathbf{J} = \mathbf{J} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{I} = 0$$

Exercise

Determine the unit base vectors in the directions of the following vectors and determine whether the vectors are orthogonal.

1 $\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$
$2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
$-2\mathbf{i} + \mathbf{j} + \mathbf{k}$

2 $2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$
$\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
$-10\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

3 $4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
$3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$
$\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

4 $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
$\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$
$6\mathbf{i} + \mathbf{j} - \mathbf{k}$

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The results are as follows:

1 $\mathbf{I} = \frac{1}{\sqrt{21}}(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}); \quad \mathbf{J} = \frac{1}{\sqrt{14}}(2\mathbf{i} + 3\mathbf{j} + \mathbf{k});$

$\mathbf{K} = \frac{1}{\sqrt{6}}(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\mathbf{I} \cdot \mathbf{J} = 0; \quad \mathbf{J} \cdot \mathbf{K} = 0; \quad \mathbf{K} \cdot \mathbf{I} = 0 \quad \therefore$ orthogonal

2 $\mathbf{I} = \frac{1}{\sqrt{17}}(2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}); \quad \mathbf{J} = \frac{1}{3}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k});$

$\mathbf{K} = \frac{1}{\sqrt{153}}(-10\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})$

$\mathbf{I} \cdot \mathbf{J} = 0; \quad \mathbf{J} \cdot \mathbf{K} = 0; \quad \mathbf{K} \cdot \mathbf{I} = 0 \quad \therefore$ orthogonal

▶

3 $\mathbf{I} = \frac{1}{\sqrt{21}}(4\mathbf{i} + 2\mathbf{j} - \mathbf{k})$; $\mathbf{J} = \frac{1}{\sqrt{38}}(3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$;

$\mathbf{K} = \frac{1}{\sqrt{41}}(\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$

$\mathbf{I} \cdot \mathbf{J} = 0$; $\mathbf{J} \cdot \mathbf{K} \neq 0$ $\quad \therefore$ not orthogonal

4 $\mathbf{I} = \frac{1}{\sqrt{14}}(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$; $\mathbf{J} = \frac{1}{\sqrt{19}}(\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$;

$\mathbf{K} = \frac{1}{\sqrt{38}}(6\mathbf{i} + \mathbf{j} - \mathbf{k})$

$\mathbf{I} \cdot \mathbf{J} = 0$; $\mathbf{J} \cdot \mathbf{K} = 0$; $\mathbf{K} \cdot \mathbf{I} \neq 0$ $\quad \therefore$ not orthogonal

Transformation equations

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In general coordinates, the transformation equations are of the form

$$x = f(u, v, w); \quad y = g(u, v, w); \quad z = h(u, v, w)$$

where the functions f, g, h are continuous and single-valued, and whose partial derivatives are continuous.

Then $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = f(u, v, w)\mathbf{i} + g(u, v, w)\mathbf{j} + h(u, v, w)\mathbf{k}$ and coordinate curves can be formed by keeping two of the three variables constant.

$$\text{Now } \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \therefore \ \mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v + \frac{\partial \mathbf{r}}{\partial w}\mathrm{d}w \qquad (1)$$

$\frac{\partial \mathbf{r}}{\partial u}$ is a tangent vector to the u-coordinate curve at P

$\frac{\partial \mathbf{r}}{\partial v}$ is a tangent vector to the v-coordinate curve at P

$\frac{\partial \mathbf{r}}{\partial w}$ is a tangent vector to the w-coordinate curve at P

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial u} \Big/ \left|\frac{\partial \mathbf{r}}{\partial u}\right| \qquad \therefore \ \frac{\partial \mathbf{r}}{\partial u} = h_u \mathbf{I} \ \text{ where } \ h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|$$

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial v} \Big/ \left|\frac{\partial \mathbf{r}}{\partial v}\right| \qquad \therefore \ \frac{\partial \mathbf{r}}{\partial v} = h_v \mathbf{J} \ \text{ where } \ h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right|$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial w} \Big/ \left|\frac{\partial \mathbf{r}}{\partial w}\right| \qquad \therefore \ \frac{\partial \mathbf{r}}{\partial w} = h_w \mathbf{K} \ \text{ where } \ h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right|$$

Then (1) above becomes

$$\mathrm{d}\mathbf{r} = h_u \mathrm{d}u \, \mathbf{I} + h_v \mathrm{d}v \, \mathbf{J} + h_w \mathrm{d}w \, \mathbf{K}$$

where, as before, h_u, h_v, h_w are the scale factors.

Element of arc ds and element of volume dV in orthogonal curvilinear coordinates

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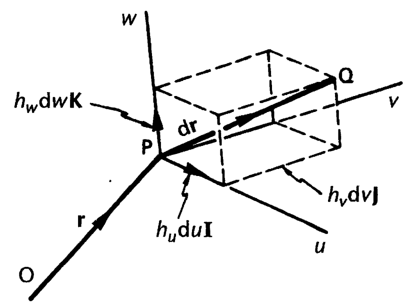

(a) *Element of arc* ds

Element of arc ds from P to Q is given by

$$\mathrm{d}\mathbf{r} = h_u\,\mathrm{d}u\,\mathbf{I} + h_v\,\mathrm{d}v\,\mathbf{J} + h_w\mathrm{d}w\,\mathbf{K}$$

$$\therefore\ \mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r} = (h_u\mathrm{d}u\,\mathbf{I} + h_v\mathrm{d}v\,\mathbf{J} + h_w\mathrm{d}w\,\mathbf{K})\cdot(h_u\mathrm{d}u\,\mathbf{I} + h_v\mathrm{d}v\,\mathbf{J} + h_w\mathrm{d}w\,\mathbf{K})$$

$$\therefore\ \mathrm{d}s^2 = h_u^2\mathrm{d}u^2 + h_v^2\mathrm{d}v^2 + h_w^2\mathrm{d}w^2$$

$$\therefore\ \mathrm{d}s = (h_u^2\mathrm{d}u^2 + h_v^2\mathrm{d}v^2 + h_w^2\mathrm{d}w^2)^{1/2}$$

(b) *Element of volume* dV

$$\mathrm{d}V = (h_u\mathrm{d}u\,\mathbf{I})\cdot(h_v\mathrm{d}v\,\mathbf{J}\times h_w\mathrm{d}w\,\mathbf{K})$$

$$= (h_u\mathrm{d}u\,\mathbf{I})\cdot(h_v\mathrm{d}v\;h_w\mathrm{d}w\,\mathbf{I}) = h_u\mathrm{d}u\;h_v\mathrm{d}v\;h_w\mathrm{d}w$$

$$\therefore\ \mathrm{d}V = h_u\,h_v\,h_w\,\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

Note also that

$$\mathrm{d}V = \left|\frac{\partial\mathbf{r}}{\partial u}\cdot\left(\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\right|\mathrm{d}u\;\mathrm{d}v\;\mathrm{d}w$$

$$= \frac{\partial(x, y, z)}{\partial(u, v, w)}\,\mathrm{d}u\;\mathrm{d}v\;\mathrm{d}w$$

where $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ is the Jacobian of the transformation.

Grad, div and curl in orthogonal curvilinear coordinates

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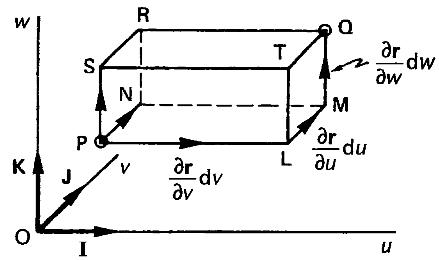
(a) *Grad V* (∇V)



Let a scalar field V exist in space and let $\mathrm{d}V$ be the change in V from P to Q. If the position vector of P is $\mathbf{r}$ then that of Q is $\mathbf{r} + d\mathbf{r}$.

Then $\mathrm{d}V = \dfrac{\partial V}{\partial u}\mathrm{d}u + \dfrac{\partial V}{\partial v}\mathrm{d}v + \dfrac{\partial V}{\partial w}\mathrm{d}w$

Let grad $V = \nabla V = (\nabla V)_u\mathbf{I} + (\nabla V)_v\mathbf{J} + (\nabla V)_w\mathbf{K}$

where $(\nabla V)_{u,v,w}$ are the components of grad V in the u, v, w directions.

Also $\quad \mathbf{dr} = \dfrac{\partial \mathbf{r}}{\partial u}\mathrm{d}u + \dfrac{\partial \mathbf{r}}{\partial v}\mathrm{d}v + \dfrac{\partial \mathbf{r}}{\partial w}\mathrm{d}w$

But $\quad \dfrac{\partial \mathbf{r}}{\partial u} = \left|\dfrac{\partial \mathbf{r}}{\partial u}\right|\mathbf{I} = h_u\,\mathbf{I}; \qquad \dfrac{\partial \mathbf{r}}{\partial v} = \left|\dfrac{\partial \mathbf{r}}{\partial v}\right|\mathbf{J} = h_v\,\mathbf{J};$

and $\quad \dfrac{\partial \mathbf{r}}{\partial w} = \left|\dfrac{\partial \mathbf{r}}{\partial w}\right|\mathbf{K} = h_w\,\mathbf{K}.$

$\therefore\ \mathbf{dr} = h_u\,\mathrm{d}u\,\mathbf{I} + h_v\,\mathrm{d}v\,\mathbf{J} + h_w\,\mathrm{d}w\,\mathbf{K}$

We have previously established that $\mathrm{d}V = \text{grad } V \cdot \mathbf{dr}$

$$\therefore\ \mathrm{d}V = \{(\nabla V)_u\mathbf{I} + (\nabla V)_v\mathbf{J} + (\nabla V)_w\mathbf{K}\} \cdot \{h_u\mathrm{d}u\mathbf{I} + h_v\mathrm{d}v\mathbf{J} + h_w\mathrm{d}w\mathbf{K}\}$$

$$= (\nabla V)_u h_u\,\mathrm{d}u + (\nabla V)_v h_v\,\mathrm{d}v + (\nabla V)_w h_w\,\mathrm{d}w$$

But $\quad \mathrm{d}V = \dfrac{\partial V}{\partial u}\mathrm{d}u + \dfrac{\partial V}{\partial v}\mathrm{d}v + \dfrac{\partial V}{\partial w}\mathrm{d}w$

▶

$\therefore$ Equating coefficients, we then have

$$\frac{\partial V}{\partial u} = (\nabla V)_u h_u \qquad \therefore\ (\nabla V)_u = \frac{1}{h_u}\frac{\partial V}{\partial u}$$

$$\frac{\partial V}{\partial v} = (\nabla V)_v h_v \qquad \therefore\ (\nabla V)_v = \frac{1}{h_v}\frac{\partial V}{\partial v}$$

$$\frac{\partial V}{\partial w} = (\nabla V)_w h_w \qquad \therefore\ (\nabla V)_w = \frac{1}{h_w}\frac{\partial V}{\partial w}$$

$$\therefore\ \text{grad } V = \nabla V = \frac{1}{h_u}\frac{\partial V}{\partial u}\mathbf{I} + \frac{1}{h_v}\frac{\partial V}{\partial v}\mathbf{J} + \frac{1}{h_w}\frac{\partial V}{\partial w}\mathbf{K}$$

$$\text{i.e. grad operator } \nabla = \frac{\mathbf{I}}{h_u}\frac{\partial}{\partial u} + \frac{\mathbf{J}}{h_v}\frac{\partial}{\partial v} + \frac{\mathbf{K}}{h_w}\frac{\partial}{\partial w}$$

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Other results we state without proof.

(b) *Div* **F** $(\nabla \cdot \mathbf{F})$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$
$$= \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right\}$$

Example 1

Show that the curvilinear expression for div **F** agrees with the earlier definition in Cartesian coordinates.

In Cartesian coordinates x, y, z we have $h_x = h_y = h_z = \ldots\ldots\ldots\ldots$ so that

$$\text{div } \mathbf{F} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{array}{l} h_x = h_y = h_z = 1 \text{ so that} \\ \text{div } \mathbf{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z} \end{array}}$$

(c) *Curl* **F** $(\nabla \times \mathbf{F})$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{I} & h_v\mathbf{J} & h_w\mathbf{K} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

Example 2

Show that the curvilinear expression for curl **F** agrees with the earlier definition in Cartesian coordinates.

In Cartesian coordinates x, y, z we have $h_x = h_y = h_z = \ldots\ldots\ldots\ldots$ and **I**, **J**, **K** $= \ldots,\ \ldots,\ \ldots$ so that

$$\text{curl } \mathbf{F} = \ldots\ldots\ldots\ldots$$

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$h_x = h_y = h_z = 1$ and $\mathbf{I}, \mathbf{J}, \mathbf{K} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ so that

$$\text{curl }\mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Because in Cartesians

$h_x = h_y = h_z = 1$ and $\mathbf{I}, \mathbf{J}, \mathbf{K} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ so that

$$\nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{I} & h_v\mathbf{J} & h_w\mathbf{K} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

$$= \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

(d) *Div grad V* $(\nabla^2 V)$

$$\text{div grad } V = \nabla \cdot (\nabla V) = \nabla^2 V$$

$$= \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\cdot\frac{\partial V}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\cdot\frac{\partial V}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\cdot\frac{\partial V}{\partial w}\right)\right\}$$

Example 3

Show that the curvilinear expression for $\nabla^2 V$ agrees with the earlier definition in Cartesian coordinates.

In Cartesian coordinates x, y, z we have $h_x = h_y = h_z = \ldots\ldots\ldots$ so that

$$\nabla^2 V = \ldots\ldots\ldots\ldots$$

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$h_x = h_y = h_z = 1$ so that

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Let's try another example, this time in coordinates other than Cartesians.

Example 4

If $V(u, v, w) = u + v^2 + w^3$ with scale factors $h_u = 2$, $h_v = 1$, $h_w = 1$, find $\nabla^2 V$ at the point (5, 3, 4).

There is very little to it. All we have to do is to determine the various partial derivatives and substitute in the expression above with relevant values.

$$\text{div grad } V = \ldots\ldots\ldots\ldots$$

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$$\boxed{26}$$

Because

$$\nabla^2 V = \frac{1}{h_u\, h_v\, h_w}\left\{\frac{\partial}{\partial u}\left(\frac{h_v\, h_w}{h_u}\cdot\frac{\partial V}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{h_u\, h_w}{h_v}\cdot\frac{\partial V}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{h_u\, h_v}{h_w}\cdot\frac{\partial V}{\partial w}\right)\right\}$$

In this case, $V = u + v^2 + w^3 \ \therefore\ \dfrac{\partial V}{\partial u} = 1;\ \ \dfrac{\partial V}{\partial v} = 2v;\ \ \dfrac{\partial V}{\partial w} = 3w^2$

Also $h_u = 2,\ h_v = 1,\ h_w = 1$

$$\therefore\ \nabla^2 V = \frac{1}{2}\left\{\frac{\partial}{\partial u}\left(\frac{1}{2}\right)+\frac{\partial}{\partial v}(4v)+\frac{\partial}{\partial w}(6w^2)\right\}$$
$$= \tfrac{1}{2}\{0 + 4 + 12w\}$$

$\therefore$ At $w = 4,\quad \nabla^2 V = 26$

That is all there is to it. Here is another.

Example 5

If $V = (u^2 + v^2)w^3$ with $h_u = 3,\ h_v = 1,\ h_w = 2$, find div grad V at the point $(2,\ -2,\ 1)$.

$$\nabla^2 V = \dots\dots\dots\dots$$

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$$\boxed{14\tfrac{2}{9}}$$

Because

$$V = (u^2 + v^2)w^3 \quad \therefore\ \frac{\partial V}{\partial u} = 2uw^3;\ \frac{\partial V}{\partial v} = 2vw^3;\ \frac{\partial V}{\partial w} = 3(u^2 + v^2)w^2$$

also $h_u = 3,\ \ h_v = 1,\ \ h_w = 2$

$$\therefore\ \nabla^2 V = \frac{1}{6}\left\{\frac{\partial}{\partial u}\left(\frac{2}{3}\frac{\partial V}{\partial u}\right)+\frac{\partial}{\partial v}\left(6\frac{\partial V}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{3}{2}\frac{\partial V}{\partial w}\right)\right\}$$
$$= \frac{1}{6}\left\{\frac{\partial}{\partial u}\left(\frac{4}{3}uw^3\right)+\frac{\partial}{\partial v}(12vw^3)+\frac{\partial}{\partial w}\left(\frac{9}{2}(u^2 + v^2)w^2\right)\right\}$$

$\therefore$ at $(2,\ -2,\ 1)$

$$\nabla^2 V = \tfrac{1}{6}\{(\tfrac{4}{3}w^3) + (12w^3) + 9(u^2 + v^2)w\}$$
$$= \tfrac{1}{6}\{\tfrac{4}{3} + 12 + 72\} = \frac{256}{18} = 14\tfrac{2}{9}$$

Particular orthogonal systems

We can apply the general results for div, grad and curl to special coordinate systems by inserting the appropriate scale factors – as we shall now see.

▶

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(a) *Cartesian rectangular coordinate system*

If we replace u, v, w by x, y, z and insert values of $h_x = h_y = h_z = 1$, we obtain expressions for grad, div and curl in rectangular coordinates, so that

grad $V = \dots\dots\dots$; div $\mathbf{F} = \dots\dots\dots$; curl $\mathbf{F} = \dots\dots\dots$

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$$\text{grad } V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}$$

$$\text{div } \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

all of which you will surely recognise.

(b) *Cylindrical polar coordinate system*

Here we simply replace u, v, w with ρ, ϕ, z and insert $h_u = h_\rho = 1$, $h_v = h_\phi = \rho$, $h_w = h_z = 1$ giving

grad $V = \dots\dots\dots$; div $\mathbf{F} = \dots\dots\dots$;
curl $\mathbf{F} = \dots\dots\dots$

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$$\text{grad } V = \frac{\partial V}{\partial \rho}\mathbf{I} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{J} + \frac{\partial V}{\partial z}\mathbf{K}$$

$$\text{div } \mathbf{F} = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \phi}(F_\phi) + \frac{\partial}{\partial z}(\rho F_z)\right\}$$

$$\text{curl } \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{I} & \rho\mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

(c) *Spherical polar coordinate system*

Replacing u, v, w with r, θ, ϕ with $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$,

grad $V = \dots\dots\dots$; div $\mathbf{F} = \dots\dots\dots$;
curl $\mathbf{F} = \dots\dots\dots$

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$$\text{grad } V = \frac{\partial V}{\partial r}\mathbf{I} + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{J} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{K}$$

$$\text{div } \mathbf{F} = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta\, F_r) + \frac{\partial}{\partial \theta}(r\sin\theta\, F_\theta) + \frac{\partial}{\partial \phi}(rF_\phi)\right\}$$

$$\text{curl } \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{I} & r\mathbf{J} & r\sin\theta\,\mathbf{K} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta\, F_\phi \end{vmatrix}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin\theta}\frac{\partial^2 V}{\partial \phi^2}$$

The results we have compiled are sometimes written in slightly different forms, but they are, of course, equivalent.

That brings us to the end of this Programme which is designed as an introduction to the topic of curvilinear coordinates. It has considerable applications, but these are beyond the scope of this present course of study.

The **Revision summary** follows as usual. Make any further notes as necessary: then you can work through the **Can You?** checklist and the **Test exercise** without difficulty. The Programme ends with the usual **Further problems**.

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Revision summary 19

1 *Curvilinear coordinates in two dimensions*

$u = f(x, y);\quad v = g(x, y)$

2 *Orthogonal coordinate system in space*

(a) *Cartesian rectangular coordinates* $(x,\ y,\ z)$

$\mathbf{F} = F_x\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$ Scale factors $h_x = h_y = h_z = 1$

(b) *Cylindrical polar coordinates* $(\rho,\ \phi,\ z)$

$\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\,\sin\phi\,\mathbf{j} + z\mathbf{k}$

Base unit vectors: Scale factors:

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial \rho}\Big/\left|\frac{\partial \mathbf{r}}{\partial \rho}\right| \qquad h_\rho = \left|\frac{\partial \mathbf{r}}{\partial \rho}\right| = 1$$

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \phi}\Big/\left|\frac{\partial \mathbf{r}}{\partial \phi}\right| \qquad h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = \rho$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial z}\Big/\left|\frac{\partial \mathbf{r}}{\partial z}\right| \qquad h_z = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1$$

$\mathbf{F} = F_\rho\,\mathbf{I} + F_\phi\,\mathbf{J} + F_z\,\mathbf{K}$

▶

(c) *Spherical polar coordinates* (r, θ, ϕ)

$$\mathbf{r} = r\sin\theta\,\cos\phi\,\mathbf{i} + r\sin\theta\,\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$$

Base unit vectors: Scale factors:

$$\mathbf{I} = \frac{\partial \mathbf{r}}{\partial r}\bigg/\left|\frac{\partial \mathbf{r}}{\partial r}\right| \qquad h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1$$

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \theta}\bigg/\left|\frac{\partial \mathbf{r}}{\partial \theta}\right| \qquad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r$$

$$\mathbf{K} = \frac{\partial \mathbf{r}}{\partial \phi}\bigg/\left|\frac{\partial \mathbf{r}}{\partial \phi}\right| \qquad h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = r\sin\theta$$

$$\mathbf{F} = F_r\,\mathbf{I} + F_\theta\,\mathbf{J} + F_\phi\,\mathbf{K}$$

3 *General orthogonal curvilinear coordinates* (u, v, w)
$x = f(u, v, w)$; $y = g(u, v, w)$; $w = h(u, v, w)$

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u} = h_u\mathbf{I} \quad \text{where} \quad h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|$$

$$\frac{\partial \mathbf{r}}{\partial v} = h_v\mathbf{J} \quad \text{where} \quad h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right|$$

$$\frac{\partial \mathbf{r}}{\partial w} = h_w\mathbf{K} \quad \text{where} \quad h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right|$$

Element of arc: $\mathrm{d}s = (h_u^2\,\mathrm{d}u^2 + h_v^2\,\mathrm{d}v^2 + h_w^2\,\mathrm{d}w^2)^{1/2}$

Element of volume: $\mathrm{d}V = h_u h_v h_w\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$

$$= \frac{\partial(x, y, z)}{\partial(u, v, w)}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

4 *Grad, div and curl in orthogonal curvilinear coordinates*

(a) Grad $V = \nabla V = \dfrac{1}{h_u}\dfrac{\partial V}{\partial u}\mathbf{I} + \dfrac{1}{h_v}\dfrac{\partial V}{\partial v}\mathbf{J} + \dfrac{1}{h_w}\dfrac{\partial V}{\partial w}\mathbf{K}$

grad operator $= \nabla = \dfrac{\mathbf{I}}{h_u}\dfrac{\partial}{\partial u} + \dfrac{\mathbf{J}}{h_v}\dfrac{\partial}{\partial v} + \dfrac{\mathbf{K}}{h_w}\dfrac{\partial}{\partial w}$

(b) Div $\mathbf{F} = \dfrac{1}{h_u h_v h_w}\left\{\dfrac{\partial}{\partial u}(h_v h_w F_u) + \dfrac{\partial}{\partial v}(h_w h_u F_v) + \dfrac{\partial}{\partial w}(h_u h_v F_w)\right\}$

(c) Curl $\mathbf{F} = \dfrac{1}{h_u h_v h_w}\begin{vmatrix} h_u\mathbf{I} & h_v\mathbf{J} & h_w\mathbf{K} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$

(d) Div grad $V = \nabla \cdot \nabla V = \nabla^2 V$

$$= \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\cdot\frac{\partial V}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\cdot\frac{\partial V}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\cdot\frac{\partial V}{\partial w}\right)\right\}$$

▶

5 *Grad, div and curl in cylindrical and spherical coordinates*

(a) *Cylindrical coordinates* (ρ, ϕ, z)

$$\text{grad } V = \frac{\partial V}{\partial \rho}\mathbf{I} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{J} + \frac{\partial V}{\partial z}\mathbf{K}$$

$$\text{div } \mathbf{F} = \frac{1}{\rho}\left\{\frac{\partial(\rho F_\rho)}{\partial \rho}\right\} + \frac{1}{\rho}\left\{\frac{\partial F_\phi}{\partial \phi}\right\} + \frac{\partial F_z}{\partial z}$$

$$\text{curl } \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{I} & \rho\mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

(b) *Spherical coordinates* (r, θ, ϕ)

$$\text{grad } V = \frac{\partial V}{\partial r}\mathbf{I} + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{J} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{K}$$

$$\text{div } \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(F_\phi)$$

$$\text{curl } \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{I} & r\mathbf{J} & r\sin\theta\,\mathbf{K} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta\, F_\phi \end{vmatrix}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Can You?

38 Checklist 19

Check this list before and after you try the end of Programme test

On a scale of 1 to 5 how confident are you that you can: **Frames**

- Derive the family of curves of constant coordinates for curvilinear coordinates? 1 to 11

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Derive unit base vectors and scale factors in orthogonal curvilinear coordinates? 12 to 24

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the element of arc ds and the element of volume dV in orthogonal curvilinear coordinates? 25

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain expressions for the operators grad, div and curl in orthogonal curvilinear coordinates? 26 to 36

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 19

39

1 Determine the unit vectors in the directions of the following three vectors and test whether they form an orthogonal set.

$$3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$
$$-2\mathbf{i} - \mathbf{j} + 4\mathbf{k}.$$

2 If $\mathbf{r} = u \sin 2\theta\, \mathbf{i} + u \cos 2\theta\, \mathbf{j} + v^2\, \mathbf{k}$, determine the scale factors h_u, h_v, h_θ.

3 If P is a point $\mathbf{r} = \rho \cos\phi\, \mathbf{i} + \rho \sin\phi\, \mathbf{j} + z\mathbf{k}$ and a scalar field $V = \rho^2 z \sin 2\phi$ exists in space, using cylindrical polar coordinates (ρ, ϕ, z) determine grad V at the point at which $\rho = 1$, $\phi = \pi/4$, $z = 2$.

4 A vector field $\mathbf{F}$ is given in cylindrical coordinates by

$$\mathbf{F} = \rho \cos\phi\, \mathbf{I} + \rho \sin 2\phi\, \mathbf{J} + z\mathbf{K}$$

Determine (a) div $\mathbf{F}$; (b) curl $\mathbf{F}$.

5 Using spherical coordinates (r, θ, ϕ) determine expressions for (a) an element of arc ds; (b) an element of volume dV.

6 If V is a scalar field such that $V = u^2 v w^3$ and scale factors are $h_u = 1$, $h_v = 2$, $h_w = 4$, determine $\nabla^2 V$ at the point $(2, 3, -1)$.

Further problems 19

40

1 Determine whether the following sets of three vectors are orthogonal.

(a) $4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ (b) $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ $4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

$\mathbf{i} - 11\mathbf{j} + 26\mathbf{k}$ $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

2 If $V(u, v, w) = v^3 w^2 \sin 2u$ with scale factors $h_u = 3$, $h_v = 1$, $h_w = 2$, determine div grad V at the point $(\pi/4, -1, 3)$.

3 A scalar field $V = \dfrac{u^2 e^{2w}}{v}$ exists in space. If the relevant scale factors are $h_u = 2$, $h_v = 3$, $h_w = 1$, determine the value of $\nabla^2 V$ at the point (1, 2, 0).

4 If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ in spherical polar coordinates (r, θ, ϕ), prove that, for any vector field $\mathbf{F}$ where

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} = F_r\mathbf{I} + F_\theta\mathbf{J} + F_\phi\mathbf{K}$$

then $F_x = F_r \sin\theta\cos\phi + F_\theta\cos\theta\cos\phi - F_\phi\sin\phi$

$$F_y = F_r\sin\theta\sin\phi + F_\theta\cos\theta\sin\phi + F_\phi\cos\phi$$

$$F_z = F_r\cos\theta - F_\theta\sin\theta.$$

5 If V is a scalar field, determine an expression for $\nabla^2 V$

(a) in cylindrical polar coordinates

(b) in spherical polar coordinates.

6 Transformation equations from rectangular coordinates (x, y, z) to parabolic cylindrical coordinates (u, v, w) are

$$x = \frac{u^2 - v^2}{2}; \quad y = uv; \quad z = w$$

V is a scalar field and $\mathbf{F}$ a vector field.

(a) Prove that the (u, v, w) system is orthogonal

(b) Determine the scale factors

(c) Find div $\mathbf{F}$

(d) Obtain an expression for $\nabla^2 V$.